Accidental degeneracy and symmetry Lie algebra

1985 J. Phys. A: Math. Gen. 18 2423

(http://iopscience.iop.org/0305-4470/18/13/018)

J. Phys. A: Math. Gen. 18 (1985) 2423-2428. Printed in Great Britain

Accidental degeneracy and symmetry Lie algebra

R Dirl† and M Moshinsky‡§

† Institut für Theoretische Physik, Technische Universitat, Wien, Austria

‡ Instituto de Física, UNAM, Apartado Postal 20-364, México, DF 01000 México

Received 4 January 1985

Abstract. In this paper we consider a very elementary example of accidental degeneracy and show that it is not explained by the symmetry Lie algebra of the corresponding classical problem.

1. Introduction

In an article published recently Moshinsky and Quesne (1983) addressed themselves to the question of whether the presence of accidental degeneracy necessarily implies the existence of a symmetry group. They did this by studying an example suggested by Konopelchenko in which the Hamiltonian was an harmonic oscillator plus a spin-orbit coupling force as well as an additional centrifugal potential. Their conclusion was that the corresponding classical problem had a well defined symmetry Lie algebra which *did not* have a quantum mechanical counterpart, and thus could not explain the accidental degeneracy of the problem.

While the Hamiltonian was very elementary, the actual analysis of the operators from which one obtained the generators of the symmetry Lie algebra in the classical limit turned out to be rather involved. Furthermore the concept of spin was used in the analysis for which one normally thinks there is no classical counterpart, though several authors (Sudashan and Mukanda 1974, Yang and Hirschfelder 1980) have proved in the last few years that this is not the case.

In view of the situation described, the present authors asked themselves whether an example of accidental degeneracy not involving the concept of spin and allowing a very simple analysis could be found, which would lead to the same conclusions as those that follow from the paper of Moshinsky and Quesne (1983). Such an example is actually available and we shall discuss it in this paper, in the hope that it will very clearly illustrate the fact that while a symmetry group may exist for the classical limit, it does not explain the accidental degeneracy of the quantum problem.

2. The problem

Let us consider the Hilbert space of square integrable functions in the unit circle characterised by the angle ϕ in the interval $0 \leqslant \phi \leqslant 2\pi$. These functions can be

§Member of the Instituto Nacional de Investigaciones Nucleares and El Colegio Nacional.

developed in terms of the kets

$$|m\rangle = (2\pi)^{-1/2} \exp(im\phi) \tag{2.1}$$

in which the latter are eigenfunctions of the angular momentum operator in the plane

$$\hat{M} = -i\,\partial/\partial\phi \tag{2.2}$$

corresponding to the eigenvalues $m = 0, \pm 1, \pm 2, \ldots$.

Let us now consider two systems associated with the angles ϕ_1, ϕ_2 and having as corresponding angular momentum operators $\hat{M}_1$, $\hat{M}_2$ defined as in (2.2). We shall now take as the Hamiltonian

$$\hat{H} = \hat{M}_1^2 + \hat{M}_2^2 \tag{2.3}$$

whose normalised eigenstates are

$$|m_1 m_2\rangle = (2\pi)^{-1} \exp[i(m_1\phi_1 + m_2\phi_2)] \tag{2.4}$$

with m_1, m_2 being arbitrary integers. The eigenvalues of $\hat{H}$ are then

$$E_{m_1 m_2} = m_1^2 + m_2^2. \tag{2.5}$$

Clearly the problem has accidental degeneracy. To begin with, there is the obvious fact that $(m_1, -m_2)$, $(-m_1, m_2)$, $(-m_1, -m_2)$, (m_2, m_1) give the same energy as (m_1, m_2). This degeneracy is due to the point group C_{4v} which is the (maximal) point group of a square lattice. There is also the possibility that a given non-negative integer can be expressed as a sum of two squares in different ways. For example, if $E = 650$ there are the possibilities

$$(m_1, m_2) = (19, 17); (23, 11); (25, 5) \tag{2.6}$$

which, together with all their permutations and changes of sign, give a total of 24 states for that energy.

In figure 1 we enumerate by the index $N = 1, 2, 3, \ldots$, in the abscissa the successive energy levels. The energies themselves are given by the upper end of the vertical lines according to the scale in the ordinate. The degeneracy is measured by the length of the line, taking as the unit the length of the first level $N = 1$, $E = 0$, which is non-degenerate. Thus, for example, $N = 2$ corresponds to $E = 1$, $(m_1, m_2) = (1, 0)$ where we write only the case when $0 \leqslant m_2 \leqslant m_1$ and its degeneracy is 4, while for $N = 14$, we have $E = 25$, $(m_1, m_2) = (4, 3)$, $(5, 0)$ and the degeneracy is 12. Note the irregular behaviour of the eigenvalues in regard to their degeneracy.

To answer the question of whether there is a symmetry group responsible for this accidental degeneracy requires first an analysis, both in classical and in quantum mechanics, of a rotor in the plane.

3. The rotor in the plane

We consider first the problem classically involving the angle ϕ and its canonically conjugate angular momentum M satisfying the Poisson bracket relation

$$\{\phi, M\} = 1 \tag{3.1}$$

and in which, as before, $0 \leqslant \phi \leqslant 2\pi$.

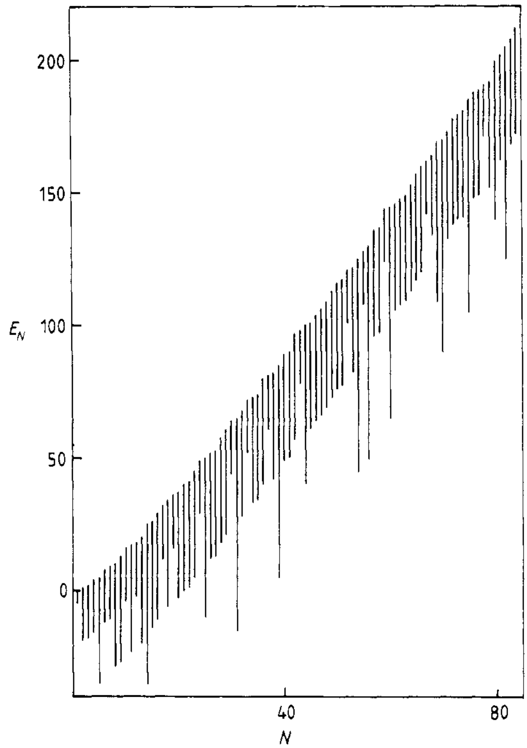

Figure 1. We enumerate by the index $N = 1, 2, 3, \ldots$, in the abscissa the successive energy levels. The energies themselves are given by the point at the upper end of the vertical lines according to the scale of the ordinate. The degeneracy is measured by the length of the line, taking as the unit the length of the first level $N = 1$, $E = 0$, which is non-degenerate.

In quantum mechanics one immediately encounters a problem in defining the multiplicative operator associated with ϕ as well as the corresponding operator of angular momentum as discussed, among others, by Levy Leblond (1976).

One could formulate the problem as follows. Let us introduce an observable χ with the standard spectrum

$$-\infty \leqslant \chi \leqslant \infty \tag{3.2}$$

and define

$$\phi = \chi - 2n\pi \qquad \text{if} \qquad 2n\pi \leqslant \chi \leqslant 2(n+1)\pi, \qquad \hat{M} = -\mathrm{i}\partial/\partial\chi \tag{3.3}$$

so that ϕ is given in figure 2.

We then immediately see from the derivative of a step function (Levy Leblond 1976) that the commutator of $\hat{M}$ and ϕ takes the value

$$[\hat{M}, \phi] = -\mathrm{i}\frac{\partial\phi}{\partial\chi} = -\mathrm{i}\left(1 - 2\pi \sum_{n=-\infty}^{\infty} \delta(\chi - 2n\pi)\right). \tag{3.4}$$

The above result can be corroborated in quantum mechanics if we take a complete set of states in the interval $0 \leqslant \chi \leqslant 2\pi$, where in this interval we replace χ by ϕ, i.e.

$$|m\rangle = (2\pi)^{-1/2} \exp(\mathrm{i}m\phi). \tag{3.5}$$

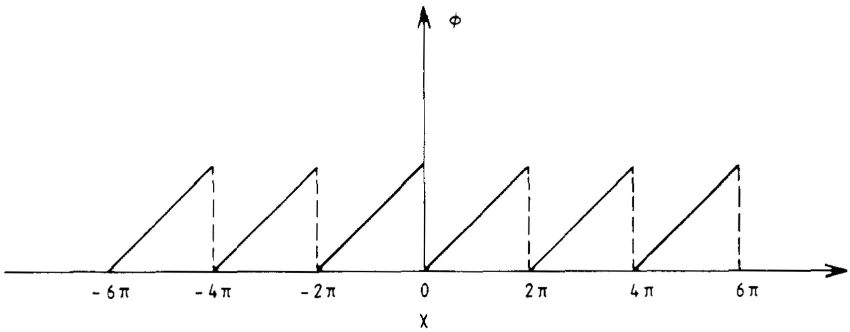

Figure 2. The periodic variable ϕ as a function of χ in the interval $-\infty \leqslant \chi \leqslant \infty$.

Then we see that

$$\langle m'|\phi|m\rangle = (2\pi)^{-1} \int_0^{2\pi} \phi \exp[\mathrm{i}(m-m')\phi]\,\mathrm{d}\phi$$

$$= \begin{cases} \mathrm{i}(m'-m)^{-1} & \text{if } m' \neq m \\ \pi & \text{if } m' = m \end{cases} \tag{3.6}$$

$$\langle m'|\hat{M}|m\rangle = m\delta_{m'm}. \tag{3.7}$$

We then have the following value for the matrix element:

$$\langle m'|[\phi, \hat{M}]|m\rangle = \sum_{m''} \{\langle m'|\phi|m''\rangle\langle m''|\hat{M}|m\rangle - \langle m'|\hat{M}|m''\rangle\langle m''|\phi|m\rangle\}$$

$$= \langle m'|\phi|m\rangle(m-m') = \begin{cases} -\mathrm{i} & \text{if } m' \neq m \\ 0 & \text{if } m' = m \end{cases} \tag{3.8}$$

which is certainly different from the $\mathrm{i}\delta_{m'm}$ we could expect if ϕ, $\hat{M}$ are considered to be canonically conjugate. On the other hand, (3.8) agrees with the matrix element on the right-hand side of (3.4).

The well known fact that the Poisson bracket $\{\phi, M\}$ does not translate into $-\mathrm{i}[\phi, \hat{M}]$ in quantum mechanics will be central to our discussion. Note, though, that the problems disappear if instead of a commutator between ϕ and $\hat{M}$, we consider those in which ϕ is replaced by a periodic function of the variable like

$$\exp(\pm\mathrm{i}\phi) = \exp(\pm\mathrm{i}\chi). \tag{3.9}$$

In this case it is clear that the Poisson brackets

$$\{M, \exp(\pm\mathrm{i}\phi)\} = \mp\mathrm{i} \exp(\pm\mathrm{i}\phi) \tag{3.10}$$

do translate into

$$[\hat{M}, \exp(\pm\mathrm{i}\phi)] = \pm\exp(\pm\mathrm{i}\phi). \tag{3.11}$$

If we now define

$$\hat{I}^{\pm} \equiv \exp(\pm\mathrm{i}\phi) \qquad \hat{I}^{0} \equiv \hat{M} = \frac{1}{\mathrm{i}}\frac{\partial}{\partial\chi} = \frac{1}{\mathrm{i}}\frac{\partial}{\partial\phi} \tag{3.12}$$

we see that these operators correspond to the Lie algebra of a Euclidean group E(2) of two dimensions

$$[\hat{I}^0, \hat{I}^{\pm}] = \pm \hat{I}^{\pm} \qquad [\hat{I}^+, \hat{I}^-] = 0 \tag{3.13}$$

which clearly is the dynamical group of the rotor in the plane.

4. The dynamical and symmetry groups of our problem

If we consider now the Hamiltonian $\hat{H} = \hat{M}_1^2 + \hat{M}_2^2$ of (2.3) it is immediately clear that its dynamical group is the direct product of two Euclidean groups

$$\mathrm{E}_1(2) \times \mathrm{E}_2(2) \tag{4.1}$$

the elements of whose Lie algebras are given by

$$\hat{I}_i^{\pm} \qquad \hat{I}_i^0 \qquad i = 1, 2 \tag{4.2}$$

defined as in (3.12) but now with indices 1, 2 on the ϕ. We immediately see that with $\hat{I}_1^{\pm}$, $\hat{I}_2^{\pm}$ we can transform any state of the type (2.4) into any other. Furthermore the Hamiltonian (2.3) is in the enveloping algebra of $\mathrm{E}_1(2) \times \mathrm{E}_2(2)$.

Our problem is to find in the enveloping algebra elements that commute with $\hat{H}$ and close under commutation among themselves, thus providing a symmetry group of $\hat{H}$. We shall proceed to show that this can be done in the classical limit but it does not have a counterpart in quantum mechanics for the reasons discussed in § 3.

Let us consider first the classical Hamiltonian

$$H = M_1^2 + M_2^2 \tag{4.3}$$

with the Poisson brackets relations

$$\{\phi_i, M_j\} = \delta_{ij} \qquad \{\phi_i, \phi_j\} = \{M_i, M_j\} = 0 \qquad i, j = 1, 2. \tag{4.4}$$

It is immediately clear that the Euclidean group whose generators are

$$M_1 \qquad M_2 \qquad K = \phi_1 M_2 - \phi_2 M_1 \tag{4.5}$$

is the symmetry group of the Hamiltonian (4.3) as $\{H, M_1\} = \{H, M_2\} = \{H, K\} = 0$ and furthermore M_1, M_2, K are the generators of an E(2) Lie algebra.

On the other hand, the corresponding quantum analysis no longer applies and in fact $\hat{K}$ is no longer an integral of motion of $\hat{H}$ as we see when we consider

$$\begin{aligned}
[\hat{K}, \hat{H}] &= [\phi_1, \hat{H}]\hat{M}_2 - [\phi_2, \hat{H}]\hat{M}_1 \\
&= \hat{M}_1[\phi_1, \hat{M}_1]\hat{M}_2 + [\phi_1, \hat{M}_1]\hat{M}_1\hat{M}_2 - \hat{M}_2[\phi_2, \hat{M}_2]\hat{M}_1 - [\phi_2, \hat{M}_2]\hat{M}_2\hat{M}_1 \\
&= -2\pi \mathrm{i}\bigg(\hat{M}_1 \sum_n \delta(\phi_1 - 2n\pi)\hat{M}_2 + \sum_n \delta(\phi_1 - 2n\pi)\hat{M}_1\hat{M}_2 \\
&\quad - \hat{M}_2 \sum_n \delta(\phi_2 - 2n\pi)\hat{M}_1 - \sum_n \delta(\phi_2 - 2n\pi)\hat{M}_2\hat{M}_1 \bigg)
\end{aligned} \tag{4.6}$$

where we made use of (3.4). Clearly then the classical symmetry group does not translate into the quantum picture and thus cannot explain the presence of an accidental degeneracy.

Another way of arriving at the same conclusion is to try to write $\hat{K} = \phi_1\hat{M}_2 - \phi_2\hat{M}_1$ in terms of the generators $\hat{I}_i^{\pm}$, $\hat{I}_i^0$, $i = 1, 2$ of the dynamical group of our problem.

Formally we could write

$$\hat{K} = -\mathrm{i}[(\ln \hat{I}_1^+)\hat{I}_2^0 - (\ln \hat{I}_2^+)\hat{I}_1^0] \tag{4.7}$$

but, because of the many-valuedness of the logarithmic function, $\hat{K}$ will not be a 'bona fide' element in the enveloping algebra of $E_1(2) \times E_2(2)$.

In conclusion, we have here a very elementary example in which one has accidental degeneracy, and in which one also has a Lie algebra classically that would explain the fact that the Hamiltonian H of (4.3) is invariant under the transformation:

$$M_1' = M_1 \cos\alpha + M_2 \sin\alpha \qquad M_2' = -M_1 \sin\alpha + M_2 \cos\alpha. \tag{4.8}$$

This Lie algebra given by (4.5) does not translate though into a corresponding one in quantum mechanics and thus does not explain the accidental degeneracy.

Acknowledgments

The authors would like to thank Professor Peter Weinberger of the Technische Universität, Wien, for his help in obtaining the data for figure 1.

References

Levy Leblond J M 1976 *Ann. Phys., NY* **101** 319
Moshinsky M and Quesne C 1983 *Ann. Phys., NY* **148** 462
Sudarshan E C G and Mukunda N 1974 *Classical Dynamics: A Modern Perspective* (New York: Wiley) pp 316–21
Yang K H and Hirschfelder J D 1980 *Phys. Rev.* A **22** 1814